On the spectra of generalized Fibonacci and Fibonacci-like operators

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Abstract

We analyze the spectra of generalized Fibonacci and Fibonacci-like operators in Banach space l^1. Some of the results have application in population dynamics.

1 Introduction and preliminaries

Let l^1 denote the Banach space of all real sequences $x \stackrel{\text{def}}{=} (x_1, x_2, x_3, \cdots)$ such that $\|x\|_1 \stackrel{\text{def}}{=} \sum |x_k| < \infty$. Let $H : l^1 \to l^1$ be a linear operator on l^1. The resolvent set of H, $\rho(H)$ is the set of all complex numbers λ such that the operator $\lambda I - H$ has a bounded inverse, where $I : l^1 \to l^1$ is the identity operator. The set $\sigma(H) \stackrel{\text{def}}{=} \mathbb{C} \setminus \rho(H)$ is the spectrum of H. The spectrum is further subdivided into three mutually disjoint parts, the point spectrum $\sigma_p(H)$, the continuous spectrum $\sigma_c(H)$ and the residual spectrum $\sigma_r(H)$. The point spectrum is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - H$ has no inverse. As in the finite dimensional case, such λ are also called eigenvalues and the corresponding non-zero vectors $x \in l^1$, such that $(\lambda I - H)x = 0$ are called eigenvectors. The continuous spectrum is the set of all λ not in $\rho(H)$ or $\sigma_p(H)$ for which the range of $\lambda I - H$ is dense in l^1. The residual spectrum is the set of all λ in $\sigma(H)$ which are not in $\sigma_p(H)$ or $\sigma_c(H)$. The spectral radius of H is

$$r_\sigma(H) \stackrel{\text{def}}{=} \sup_{\lambda \in \sigma(H)} |\lambda|. \tag{1}$$

The operator H has a matrix representation $\mathbf{H}$ in the standard basis $\mathbf{e}_{ik} \stackrel{\text{def}}{=} \delta_{ik}$, where δ_{ik} is the Kronecker symbol.

We shall also use two standard results: first, if the operator H is bounded or closed and has a matrix representation $\mathbf{H}$, then the transpose matrix $\mathbf{H}^t$ is the matrix representation of the operator $H^t : l^\infty \to l^\infty$ and (see e.g. [TaLa80], [Gol66, Corollary II.5.3] or [Hal55, Theorems 3.2 and 3.3])

$$\sigma_p(H^t) \subseteq \sigma_p(H) \cup \sigma_r(H), \quad \sigma_r(H) \subseteq \sigma_p(H^t). \tag{2}$$

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Second, if H is bounded, then (see for example [TaLa80, (3-5)])

$$r_\sigma(H) = \lim_{k\to\infty} \|H^k\|_1^{1/k}. \tag{3}$$

Our aim is to classify spectra of two classes of generalized Fibonacci and Fibonacci-like operators. For the first class of operators their spectral radii are expressed in terms of largest real positive roots of certain polynomials and the coefficients of their powers behave like generalized Fibonacci sequences, as we shall see in section 2.

The second class of operators, which also has applications in mathematical biology, is analyzed in a similar manner in section 3.

2 Generalized Fibonacci operators

Let the linear operator $F_n : l^1 \to l^1$ be defined by

$$(x_1, x_2, x_3, \cdots) \to \left(\sum_{k=n+1}^{\infty} x_k, x_1, x_2, x_3, \cdots \right), \qquad n = 1, 2, 3, \ldots \tag{4}$$

Each F_n is bounded and its matrix representation in the standard basis is

$$\mathbf{F}_n = \begin{array}{l} \quad\overbrace{\hphantom{0 \quad 0 \quad \cdots \quad 0}}^{n} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}, \tag{5}$$

Following the analysis of the spectrum of F_1 by Halberg [Hal55], the spectrum of F_n is classified in several steps which are summarized as follows:

1. first, by solving the equation

$$(\lambda I - F_n)\, x = 0, \quad x \neq 0, \tag{6}$$

we show that the point spectrum is

$$\sigma_p(F_n) = \{\lambda \in \mathbb{C} : \ \lambda^{n+1} - \lambda^n - 1 = 0, \ |\lambda| > 1\}, \tag{7}$$

2. second, by solving the equation

$$(\lambda I - F_n)\, x = y, \quad x \neq 0, \tag{8}$$

we compute the inverse $(\lambda I - F_n)^{-1}$ and show that the resolvent set consists of all λ such that $|\lambda| > 1$ which are not in $\sigma_p(F_n)$, that is,

$$\rho(F_n) = \{\lambda \in \mathbb{C} : \ |\lambda| > 1, \ \lambda^{n+1} - \lambda^n - 1 \neq 0\}, \tag{9}$$

3. third, we analyze the transposed operator F_n^t and show that

$$\sigma_p(F_n^t) = \{\lambda \in \mathbb{C} : |\lambda| \le 1,\ \lambda \ne 1\}, \tag{10}$$

which, together with (2), implies that the residual spectrum of F_n is

$$\sigma_r(F_n) = \{\lambda \in \mathbb{C} :\ |\lambda| \le 1,\ \lambda \ne 1\}. \tag{11}$$

4. Finally, since the spectrum of F_n is closed, is also contains the point $\lambda = 1$. Since this point is neither in the point spectrum nor in the residual spectrum, it must be in the continuous spectrum, that is

$$\sigma_c(F_n) = \{1\}. \tag{12}$$

We proceed with the detailed analysis of each step.
Step 1. The equation (6) can be written as

$$\begin{aligned} 0 &= \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots, \\ x_1 &= \lambda x_2, \\ x_2 &= \lambda x_3, \\ &\vdots \\ x_k &= \lambda x_{k+1}, \\ &\vdots \end{aligned} \tag{13}$$

Since $\lambda = 0$ implies $x = 0$, zero is not an element of $\sigma_p(F_n)$. If $\lambda \ne 0$, by applying (13) recursively, we have

$$x_{k+1} = \frac{1}{\lambda} x_k = \frac{1}{\lambda^2} x_{k-1} = \frac{1}{\lambda^3} x_{k-2} = \cdots = \frac{1}{\lambda^k} x_1, \qquad k \ge 1. \tag{14}$$

Thus

$$x = x_1 \begin{pmatrix} 1 & \frac{1}{\lambda} & \frac{1}{\lambda^2} & \cdots & \frac{1}{\lambda^k} & \cdots \end{pmatrix}^t \tag{15}$$

and

$$\|x\|_1 = |x_1| \sum \frac{1}{|\lambda|^k}. \tag{16}$$

If $|\lambda| \le 1$, then $\|x\|_1 = \infty$, so $x \notin l^1$. If $|\lambda| > 1$, then $\|x\|_1 = |x_1|\,|\lambda|/(|\lambda| - 1)$. Inserting (14) into the first equality of (13) gives

$$\begin{aligned} 0 &= \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots \\ &= \lambda\, x_1 - \frac{1}{\lambda^n} x_1 - \frac{1}{\lambda^{n+1}} x_1 - \frac{1}{\lambda^{n+2}} x_1 - \cdots \\ &= x_1 \left[\lambda - \frac{1}{\lambda^n} \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3} + \cdots \right) \right] \\ &= x_1 \left(\lambda - \frac{1}{\lambda^n} \frac{1}{1 - \frac{1}{\lambda}} \right) \\ &= x_1 \frac{\lambda^{n+1} - \lambda^n - 1}{\lambda^{n-1}(\lambda - 1)}. \end{aligned}$$

Since $x_1 \neq 0$, we conclude that $\sigma_p(F_n)$ consists of those roots of the polynomial

$$p_{n+1}(\lambda) \stackrel{\text{def}}{=} \lambda^{n+1} - \lambda^n - 1 \tag{17}$$

for which $|\lambda| > 1$, as stated in (7).[1]

Since $p_{n+1}(1) = -1 < 0$ and $p'_{n+1}(\lambda) > 0$ for $\lambda \in \mathbb{R}, \lambda \geq 1$, that is, p_{n+1} is strictly increasing for $\lambda > 1$, we conclude that F_n has exactly one real eigenvalue larger than one. Let us denote this eigenvalue by $\lambda_{\max}(F_n)$. By Ostrovsky's theorem [Pra04, Theorem 1.1.4, p. 3], $\lambda_{\max}(F_n)$ is the unique positive root of $p_{n+1}(\lambda)$ and the absolute values of all other roots are strictly smaller. Consequently, all other eigenvalues of F_n are in absolute value strictly smaller than $\lambda_{\max}(F_n)$ which, in turn, implies

$$r_\sigma(F_n) = \lambda_{\max}(F_n). \tag{18}$$

Figure 1 shows $\sigma_p(F_n)$ for various values of n.

Step 2. The equation (8) can be written as

$$\begin{aligned} y_1 &= \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots , \\ x_2 &= \frac{1}{\lambda}(x_1 + y_2), \\ x_3 &= \frac{1}{\lambda}(x_2 + y_3), \\ &\vdots \\ x_{k+1} &= \frac{1}{\lambda}(x_k + y_{k+1}), \\ &\vdots \end{aligned} \tag{19}$$

By setting

$$u = \sum_{k=n+1}^{\infty} x_k, \qquad v = \sum_{k=n+1}^{\infty} y_k,$$

and using (19), we have

$$u = \frac{1}{\lambda}x_n + \frac{1}{\lambda}u + \frac{1}{\lambda}v, \qquad y_1 = \lambda x_1 - u.$$

After rearranging, we have

$$u = \frac{1}{\lambda - 1}(x_n + v).$$

Thus,

$$y_1 = \lambda x_1 - \frac{1}{\lambda - 1}(x_n + v). \tag{20}$$

[1] These roots are the eigenvalues and the vectors x defined by (15) are the corresponding eigenvectors.

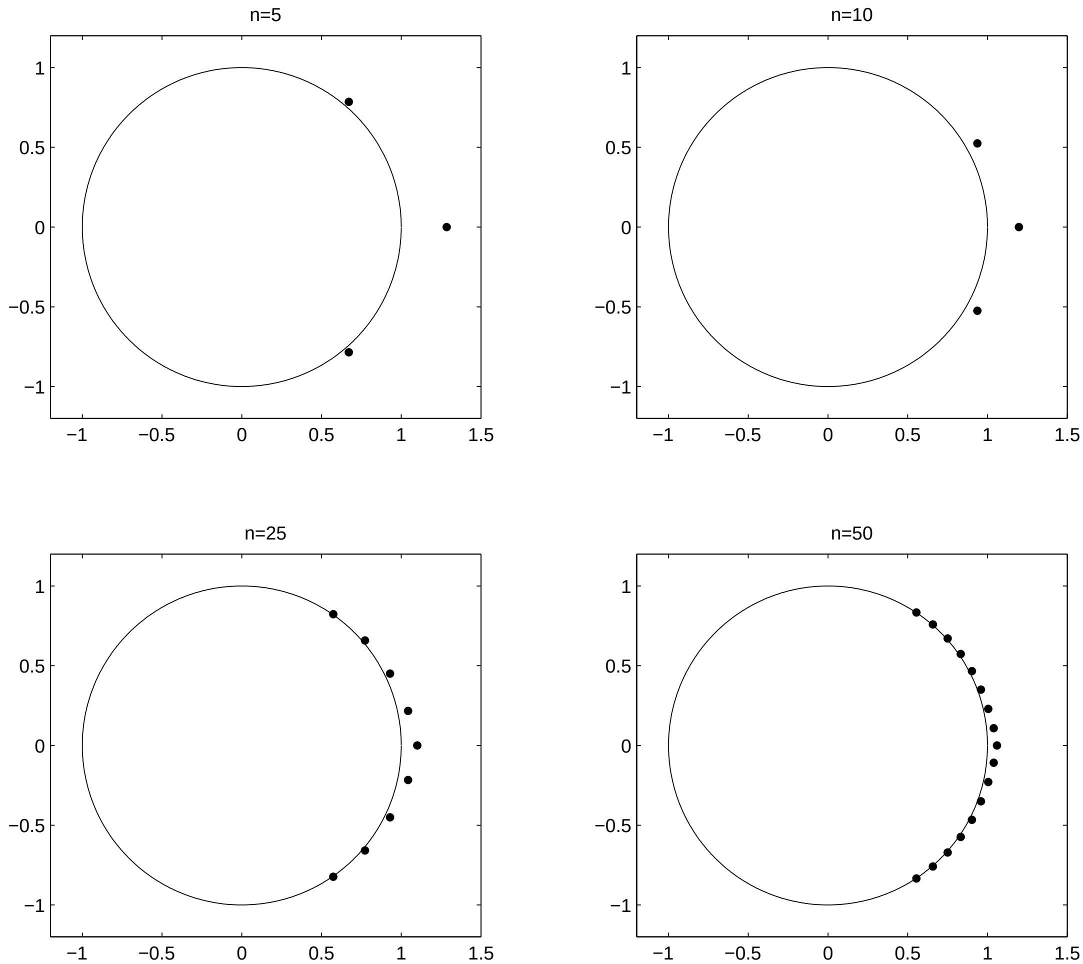

Figure 1: The point spectra $\sigma_p(F_n)$ for various values of n.

By recursively applying (19), we have

$$
\begin{aligned}
x_2 &= \frac{1}{\lambda}\,(x_1 + y_2),\\
x_3 &= \frac{1}{\lambda}\,(x_2 + y_3) = \frac{1}{\lambda^2}\,x_1 + \frac{1}{\lambda^2}\,y_2 + \frac{1}{\lambda}\,y_3,\\
&\ \vdots\\
x_{k+1} &= \frac{1}{\lambda}\,(x_k + y_{k+1}) = \frac{1}{\lambda^k}\,x_1 + \frac{1}{\lambda^k}\,y_2 + \frac{1}{\lambda^{k-1}}\,y_3 + \frac{1}{\lambda^{k-2}}\,y_4 + \cdots + \frac{1}{\lambda}\,y_{k+1}, \qquad (21)\\
&\ \vdots
\end{aligned}
$$

Inserting x_n into (20) gives

$$y_1 = \lambda\,x_1 - \frac{1}{\lambda - 1}\left(\frac{1}{\lambda^{n-1}}\,x_1 + \frac{1}{\lambda^{n-1}}\,y_2 + \frac{1}{\lambda^{n-2}}\,y_3 + \cdots + \frac{1}{\lambda}\,y_n + v\right),$$

and solving for x_1 gives

$$x_1 = \frac{1}{\lambda^{n+1} - \lambda^n - 1}\left(\lambda^{n-1}(\lambda-1)\,y_1 + y_2 + \lambda\,y_3 + \lambda^2\,y_4 + \cdots + \lambda^{n-2}\,y_n + \lambda^{n-1}\,v\right).$$

By inserting this into (21) we have

$$x = (\lambda I - F_n)^{-1} y = \frac{1}{\lambda^{n+1} - \lambda^n - 1}\,(A+B)\,y$$

where the matrix representations of A and B are given by[2]

$$\mathbf{A} = \begin{pmatrix} (\lambda-1)\lambda^{n-1} & 1 & \lambda & \lambda^2 & \lambda^3 & \cdots & \lambda^{n-2} & \lambda^{n-1} & \lambda^{n-1} & \cdots \\ (\lambda-1)\lambda^{n-2} & \frac{1}{\lambda} & 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-3} & \lambda^{n-2} & \lambda^{n-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ (\lambda-1)\lambda & \frac{1}{\lambda^{n-2}} & \frac{1}{\lambda^{n-3}} & \frac{1}{\lambda^{n-4}} & \cdots & \frac{1}{\lambda} & 1 & \lambda & \lambda & \cdots \\ (\lambda-1) & \frac{1}{\lambda^{n-1}} & \frac{1}{\lambda^{n-2}} & \frac{1}{\lambda^{n-3}} & \cdots & \frac{1}{\lambda^2} & \frac{1}{\lambda} & 1 & 1 & \cdots \\ (\lambda-1)\,\frac{1}{\lambda} & \frac{1}{\lambda^{n}} & \frac{1}{\lambda^{n-1}} & \frac{1}{\lambda^{n-2}} & \cdots & \frac{1}{\lambda^3} & \frac{1}{\lambda^2} & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{\lambda} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{\lambda^2} & \frac{1}{\lambda} & 0 & 0 & \cdots \\ 0 & \frac{1}{\lambda^3} & \frac{1}{\lambda^2} & \frac{1}{\lambda} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

respectively. Obviously, for $|\lambda| > 1$ we have $\|A\|_1 < \infty$ and $\|B\|_1 < \infty$. Thus, for $|\lambda| > 1$ and λ not being the root of $\lambda^{n+1} - \lambda^n - 1$, the operator $\lambda I - F_n$ has a bounded inverse, so the resolvent set of F_n is given by (9).

Step 3. The point spectrum of the transposed operator F_n^t consists of all $\lambda \in \mathbb{R}$ such that

$$(\lambda I - F_n^t)\,x = 0, \qquad x \neq 0, \quad \|x\|_\infty < \infty.$$

This is equivalent to

$$\begin{aligned} x_2 &= \lambda\,x_1, \\ x_3 &= \lambda\,x_2 = \lambda^2\,x_1, \\ &\vdots \\ x_{n+1} &= \lambda\,x_n = \lambda^n\,x_1, \\ x_{n+2} &= \lambda\,x_{n+1} - x_1 = (\lambda^{n+1} - 1)\,x_1, \\ x_{n+3} &= \lambda\,x_{n+2} - x_1 = (\lambda^{n+2} - \lambda - 1)\,x_1, \\ &\vdots \\ x_k &= (\lambda^{k-1} - \lambda^{k-n-2} - \lambda^{k-n-3} - \cdots - \lambda - 1)\,x_1, \\ &\vdots \end{aligned}$$

[2] Next row of $\mathbf{A}$ is obtained by dividing the previous row by λ.

Therefore,

$$x_k = \left(\lambda^{k-1} - \frac{\lambda^{k-n-1} - 1}{\lambda - 1}\right) x_1.$$

For $|\lambda| \leq 1$, $\lambda \neq 1$ we have

$$|x_k| < \left(1 + \frac{2}{|\lambda - 1|}\right) |x_1|,$$

which implies $\|x\|_\infty < \infty$. For $\lambda = 1$ we have

$$\begin{aligned}
x_2 &= x_1, \\
x_3 &= x_1, \\
&\vdots \\
x_{n+1} &= x_1, \\
x_{n+2} &= 0, \\
x_{n+3} &= -x_1, \\
x_{n+4} &= -2\,x_1, \\
&\vdots \\
x_k &= -(k-n-2)\,x_1, \\
&\vdots
\end{aligned}$$

so $\|x\|_\infty = \infty$. We conclude that the point spectrum of F_n^t is given by (10). This, in turn, implies (11) and (12) as described before.

2.1 Relationship to generalized Fibonacci sequences

In this section we describe the relationship between operators F_n and generalized Fibonacci sequences. A generalized Fibonacci sequence $\{f^{(n)}\}$ is defined by

$$f_1^{(n)} = 1, \quad f_2^{(n)} = 1, \cdots, f_{n+1}^{(n)} = 1, \quad f_k^{(n)} = f_{k-1}^{(n)} + f_{k-n-1}^{(n)}, \qquad k > n+1. \tag{22}$$

For $n = 1$ this definition yields the classical Fibonacci sequence

$$f_1 = 1, \quad f_2 = 1, \quad f_k = f_{k-1} + f_{k-2}, \quad k > 2. \tag{23}$$

By induction we can prove that the k-th power of the matrix $\mathbf{F}_n$ from (5) for $k > n$ has the form

$$\mathbf{F}_n^k = \begin{pmatrix} f_{k-n}^{(n)} & f_{k-n+1}^{(n)} & f_{k-n+2}^{(n)} & \cdots & f_{k-1}^{(n)} & f_k^{(n)} & f_k^{(n)} & f_k^{(n)} & \cdots \\ f_{k-n-1}^{(n)} & f_{k-n}^{(n)} & f_{k-n+1}^{(n)} & \cdots & f_{k-2}^{(n)} & f_{k-1}^{(n)} & f_{k-1}^{(n)} & f_{k-1}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ f_1^{(n)} & f_2^{(n)} & f_3^{(n)} & \cdots & f_n^{(n)} & f_{n+1}^{(n)} & f_{n+1}^{(n)} & f_{n+1}^{(n)} & \cdots \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

We conclude that

$$\|F_n^k\|_1 = 1 + \sum_{i=1}^{k} f_i^{(n)}. \tag{24}$$

By applying (22) to the terms in parentheses we have

$$\begin{aligned} 2\sum_{i=1}^{k} f_i^{(n)} &= f_1^{(n)} + \cdots + f_n^{(n)} + \left(f_{n+1}^{(n)} + f_{n+2}^{(n)} + \cdots + f_{k-1}^{(n)}\right) + f_k^{(n)} \\ &\quad + \left(f_1^{(n)} + f_2^{(n)} + \cdots + f_{k-n-1}^{(n)}\right) + f_{k-n}^{(n)} + f_{k-n+1}^{(n)} + \cdots + f_k^{(n)} \\ &= f_1^{(n)} + f_2^{(n)} + \cdots + f_n^{(n)} + f_{n+2}^{(n)} + \cdots + f_k^{(n)} \\ &\quad + f_k^{(n)} + f_{k-n}^{(n)} + f_{k-n+1}^{(n)} + \cdots + f_k^{(n)} \\ &= \sum_{i=1}^{k} f_i^{(n)} - f_{n+1}^{(n)} + f_{k-n}^{(n)} + f_{k-n+1}^{(n)} + \cdots + f_k^{(n)} + f_k^{(n)}. \end{aligned}$$

From this, by applying (22) again recursively, we obtain

$$\begin{aligned} \sum_{i=1}^{k} f_i^{(n)} &= f_{k-n}^{(n)} + f_{k-n+1}^{(n)} + \cdots + f_k^{(n)} + f_k^{(n)} - 1 \\ &= f_{k-n+1}^{(n)} + \cdots + f_k^{(n)} + f_{k+1}^{(n)} - 1 \\ &= f_{k-n+2}^{(n)} + \cdots + f_{k+1}^{(n)} + f_{k+2}^{(n)} - 1 \\ &\vdots \\ &= f_{k+n+1}^{(n)} - 1. \end{aligned}$$

Inserting this into (24) gives

$$\|F_n^k\|_1 = f_{k+n+1}^{(n)} \tag{25}$$

and from (18) it follows that

$$\lim_{k\to\infty} \left(f^{(n)}_{k+n+1}\right)^{1/k} = \lambda_{\max}(F_n).$$

Also, by using standard techniques in analyzing linear recurrence relations with constant coefficients, we can prove that for all i, j [3]

$$\lim_{k\to\infty} \frac{[\mathbf{F}^k_n]_{i,j}}{[\mathbf{F}^k_n]_{i+1,j}} \equiv \lim_{m\to\infty} \frac{f^{(n)}_{m+1}}{f^{(n)}_m} = \lambda_{\max}(F_n).$$

For example, by setting $n = 1$ we have for the Fibonacci sequence (23)

$$\lim_{k\to\infty} (f_{k+2})^{1/k} = r_\sigma(F_1) = \frac{1+\sqrt{5}}{2},$$
$$\lim_{k\to\infty} \frac{f_{k+1}}{f_k} = \frac{1+\sqrt{5}}{2}.$$

3 Fibonacci-like operators

Now we would like to consider the family of linear operators $\Gamma_n : l^1 \to l^1$ defined by

$$(x_1, x_2, x_3, \cdots) \to \left(\rho \sum_{k=n+1}^{\infty} (k-n)\, x_k, x_1, x_2, x_3, \cdots \right), \qquad n = 1, 2, 3, \ldots \tag{26}$$

for some real positive ρ. The domain of Γ_n is

$$\text{Dom } \Gamma_n = \left\{ x \in l^1 : \left| \sum_{k=n+1}^{\infty} (k-n) x_k \right| < \infty \right\},$$

and its matrix representation in the standard basis is

$$\mathbf{\Gamma}_n = \begin{array}{c} \overbrace{\hphantom{0 \quad 0 \quad \cdots \quad 0}}^{n} \hphantom{\quad \rho \quad 2\rho \quad 3\rho \quad 4\rho \quad 5\rho \quad \cdots} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & \rho & 2\rho & 3\rho & 4\rho & 5\rho & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}. \tag{27}$$

However, the operator Γ_n is not closed as illustrated by the following example.

[3]The proof is derived using the fact that $f^{(n)}_l$ has the form $f^{(n)}_l = \alpha\, \lambda^l_{\max}(F_n) + \sum_{i=1}^n \alpha_i \lambda_i^l$, where $\lambda_{\max}(F_n)$ and λ_i are the roots of the characteristic polynomial (17), and $|\lambda_{\max}(F_n)| > |\lambda_i|$.

Example 1 Let us define the sequence $\{x^{(m)}\}$ of vectors in l^1 by

$$x^{(m)} = \Big(\overbrace{0 \quad \cdots \quad 0}^{m+n-1} \quad \frac{1}{m} \quad 0 \quad \cdots\Big)^t .$$

Then

$$x^{(m)} \to \begin{pmatrix}0 & 0 & \cdots\end{pmatrix}^t ,$$

while

$$\mathbf{\Gamma}_n \, x^{(m)} = \begin{pmatrix}\rho & 0 & \cdots & 0 & \frac{1}{m} & 0 & \cdots\end{pmatrix}^t \to \begin{pmatrix}\rho & 0, & 0 & \cdots\end{pmatrix}^t .$$

Although the point spectrum of Γ_n is defined and can be computed in a standard manner (see later), the resolvent set of Γ_n is empty, which makes the analysis of Γ_n less interesting. Instead, we shall consider the family of operators $G_n : l_1 \to l_1$ formally defined by

$$G_n = D_n \Gamma_n D_n^{-1},$$

where

$$D_n = \mathrm{diag}\big(\overbrace{1, \cdots, 1}^{n}, \rho, 2\rho, 3\rho, 4\rho, \cdots\big).$$

That is, for $n \in \mathbb{N}$ the operator G_n is defined by

$$(x_1, x_2, x_3, \cdots) \to \left(\sum_{k=n+1}^{\infty} x_k, x_1, x_2, \cdots, x_{n-1}, \rho\, x_n, 2\, x_{n+1}, \frac{3}{2}\, x_{n+2}, \frac{4}{3}\, x_{n+3}, \frac{5}{4}\, x_{n+4}, \cdots\right),$$

and its matrix representation in the standard basis is

$$\mathbf{G}_n = \begin{array}{c} \overbrace{\hphantom{0 \quad 0 \quad \cdots \quad 0}}^{n} \hphantom{\quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots} \\ \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
0 & 0 & \cdots & \rho & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 2 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \frac{4}{3} & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix} \end{array} . \tag{28}$$

Let us define the polynomial $q_{n+1}(\lambda)$ by

$$q_{n+1}(\lambda) = \lambda^{n+1} - 2\,\lambda^n + \lambda^{n-1} - \rho. \tag{29}$$

Similarly as in section 2, the spectrum of G_n is classified in several steps which are summarized as follows:

1. first, by solving the equation

$$(\lambda I - G_n)\, x = 0, \quad x \neq 0, \tag{30}$$

we show that the point spectrum is

$$\sigma_p(G_n) = \{\lambda \in \mathbb{C} : \; q_{n+1}(\lambda) = 0, \; |\lambda| > 1\}, \quad n \geq 2. \tag{31}$$

2. second, by solving the equation

$$(\lambda I - G_n)\, x = y, \quad x \neq 0, \tag{32}$$

we compute the inverse $(\lambda I - G_n)^{-1}$ and show that the resolvent set consists of all λ such that $|\lambda| > 1$ which are not in $\sigma_p(G_n)$,

$$\rho(G_n) = \{\lambda \in \mathbb{C} : \; |\lambda| > 1, \; \lambda \notin \sigma_p(G_n)\}, \tag{33}$$

3. third, we analyze the transposed operator G_n^t and show that

$$\sigma_p(G_n^t) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1, \; \lambda \neq 1\}, \tag{34}$$

which, together with (2), implies that the residual spectrum of G_n is

$$\sigma_r(G_n) = \{\lambda \in \mathbb{C} : \; |\lambda| \leq 1, \; \lambda \neq 1\}. \tag{35}$$

4. Finally, since the spectrum of G_n is closed, is also contains the point $\lambda = 1$. Since this point is neither in the point spectrum nor in the residual spectrum, it must be in the continuous spectrum, that is

$$\sigma_c(G_n) = \{1\}. \tag{36}$$

The proofs are similar to the ones from section 2, but more tedious.

Step 1. The equation (30) can be written as

$$\begin{aligned} 0 &= \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots , \\ x_k &= \lambda x_{k+1}, \qquad k = 1, \cdots, n-1, \\ \rho\, x_n &= \lambda\, x_{n+1}, \\ \frac{k-n+1}{k-n}\, x_k &= \lambda\, x_{k+1}, \qquad k = n+1, n+2, \cdots . \end{aligned} \tag{37}$$

Since $\lambda = 0$ implies $x = 0$, zero is not an element of $\sigma_p(G_n)$. If $\lambda \neq 0$, by applying (37) recursively, we obtain

$$\begin{aligned} x_k &= \frac{1}{\lambda^{k-1}}\, x_1, \qquad k = 2, 3, \cdots, n, \\ x_k &= \rho\, \frac{k-n}{\lambda^{k-1}}\, x_1, \qquad k = n+1, n+2, \cdots . \end{aligned} \tag{38}$$

This, in turn, implies

$$x = x_1 \begin{pmatrix} 1 & \frac{1}{\lambda} & \frac{1}{\lambda^2} & \cdots & \frac{1}{\lambda^{n-1}} & \frac{\rho}{\lambda^n} & \frac{2\rho}{\lambda^{n+1}} & \frac{3\rho}{\lambda^{n+2}} & \cdots & \frac{(k-n)\rho}{\lambda^{k-1}} & \cdots \end{pmatrix}^t,$$

so that

$$\|x\|_1 = |x_1| \left(\frac{1-(\frac{1}{\lambda})^n}{1-\frac{1}{\lambda}} + \frac{\rho}{\lambda^n}(1 + \frac{2}{\lambda} + \frac{3}{\lambda^2} + \cdots) \right).$$

If $|\lambda| \leq 1$, then $\|x\|_1 = \infty$, so $x \notin l^1$. If $|\lambda| > 1$, then, by using differentiation of the geometric series, we have

$$\|x\|_1 = |x_1| \left(\frac{\lambda^n - 1}{\lambda^{n-1}(\lambda - 1)} + \frac{\rho}{\lambda^n} \frac{\lambda^2}{(\lambda-1)^2} \right) < \infty,$$

thus, $x \in l^1$. By inserting (38) into the first equality of (37) and using differentiation of the geometric series we have

$$\begin{aligned} 0 &= \lambda\, x_1 - \rho \left(\frac{1}{\lambda^n}\, x_1 + \frac{2}{\lambda^{n+1}}\, x_1 + \frac{3}{\lambda^{n+2}}\, x_1 + \cdots \right) \\ &= x_1 \left[\lambda - \frac{\rho}{\lambda^n} \left(1 + \frac{2}{\lambda} + \frac{3}{\lambda^2} + \frac{4}{\lambda^3} + \cdots \right) \right] \\ &= x_1 \left(\lambda - \frac{\rho}{\lambda^n} \frac{1}{(1-\frac{1}{\lambda})^2} \right). \end{aligned}$$

Finally, solving this equation with $x_1 \neq 0$ gives (31).

We shall now prove that $\sigma_p(G_n)$ consists of $\lambda_{\max}(G_n)$, a unique simple real eigenvalue larger than one and all other eigenvalues have modulus smaller than $\lambda_{\max}(G_n)$. This also implies

$$r_\sigma(G_n) = \lambda_{\max}(G_n). \tag{39}$$

The proof is based on the ideas from the proof of [Pra04, Theorem 1.1.4, pp. 3-4]. Indeed, if $n = 1$ then the roots of $q_2(\lambda)$ are $1 \pm \sqrt{\rho}$ and the statement holds. For $n \geq 2$ we have

$$q_{n+1}(\lambda) = \lambda^{n-1}(\lambda-1)^2 - \rho, \tag{40}$$

$$q'_{n+1}(\lambda) = \lambda^{n-2}[(n+1)\lambda^2 - 2n\lambda + (n-1)]. \tag{41}$$

Since $q_{n+1}(1) = -\rho < 0$ and $q'_{n+1}(\lambda) > 0$ for $\lambda \in \mathbb{R}, \lambda > 1$, that is, $q_{n+1}(\lambda)$ is strictly increasing for $\lambda > 1$, we conclude that $q_{n+1}(\lambda)$ has exactly one real root larger than one or, equivalently, that G_n has exactly one real eigenvalue larger than one. Let us denote this eigenvalue by $\lambda_{\max}(G_n)$. Let $z \neq \lambda_{\max}(G_n)$ be some other real or complex eigenvalue of G_n and let $\zeta = |z| > 1$. Since z is also the root of $q_{n+1}(\lambda)$, the relation (40) implies

$$z^{n-1}(z-1)^2 = \rho,$$

which, in turn, implies

$$|z|^{n-1}|z-1|^2 = \rho. \tag{42}$$

Since $\zeta > 1$, this implies

$$\zeta^{n-1}(\zeta-1)^2 \leq \rho,$$

or

$$q_{n+1}(\zeta) = \zeta^{n-1}(\zeta-1)^2 - \rho \le 0. \tag{43}$$

Since $q_{n+1}(\lambda)$ is strictly increasing for $\lambda > 1$, and $q_{n+1}(\lambda_{\max}(G_n)) = 0$, we conclude that $\zeta \le \lambda_{\max}(G_n)$ and that the equality in (43) holds only if $\zeta = \lambda_{\max}(G_n)$. But, the equality in (43) and (42) imply

$$|z-1| = \zeta - 1,$$

that is, $z \in \mathbb{R}$ and $z = \pm\zeta = \pm\lambda_{\max}(G_n)$. The choice $z = -\lambda_{\max}(G_n)$ is impossible since $q_{n+1}(-\lambda_{\max}(G_n)) \ne 0$, and the second choice contradicts the assumption $z \ne \lambda_{\max}(G_n)$. Therefore, $\zeta < \lambda_{\max}(G_n)$ as desired.

Remark 1 Although the above analysis is sufficient for our purposes, by inspecting the polynomial $q_{n+1}(\lambda)$ and its derivative from (40) and (41), respectively, we can establish further facts about its roots. From (41) we see that the derivative $q'_{n+1}(\lambda)$ has exactly two real positive simple roots $\lambda_1 = \frac{n-1}{n+1}$ and $\lambda_2 = 1$ and, if $n > 2$, also the root $\lambda_0 = 0$. If $n > 3$ then λ_0 is multiple. Let $\rho_0 = 4\lambda_1^{n-1}/(n+1)^2$. The number of real roots of $q_{n+1}(\lambda)$ in the open interval $(0, \lambda_{\max}(G_n))$ is governed by ρ as follows: if $\rho > \rho_0$, then there are no such roots, if $\rho = \rho_0$ there is exactly one root equal to λ_1 and if $\rho < \rho_0$ there are exactly two roots, one smaller and one larger than λ_1. Finally, if n is odd, then $q_{n+1}(\lambda)$ also has a simple negative real root. As we have already proved, $\lambda_{\max}(G_n)$ is the root with strictly maximal absolute value.

Remark 2 It is easy to see that the point spectrum of Γ_n from (26) and (27) is equal to the point spectrum of G_n.

Step 2. The equation (32) can be written as

$$\begin{aligned}
y_1 &= \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots, \\
x_{k+1} &= \frac{1}{\lambda}(x_k + y_{k+1}), \qquad k = 1, \cdots, n-1, \\
x_{n+1} &= \frac{1}{\lambda}(\rho\, x_n + y_{n+1}), \\
x_{k+1} &= \frac{1}{\lambda}\left(\frac{k-n+1}{k-n}\, x_k + y_{k+1}\right), \qquad k = n+1, n+2, \cdots.
\end{aligned} \tag{44}$$

By recursively applying the above three equalities, we obtain

$$\begin{aligned}
x_k = {} & \frac{(k-n)\,\rho}{\lambda^{k-1}}\, x_1 + \frac{(k-n)\,\rho}{\lambda^{k-1}}\, y_2 + \frac{(k-n)\,\rho}{\lambda^{k-2}}\, y_3 + \cdots + \frac{(k-n)\,\rho}{\lambda^{k-n+1}}\, y_n + \\
& + \frac{k-n}{1}\, \frac{1}{\lambda^{k-n}}\, y_{n+1} + \frac{k-n}{2}\, \frac{1}{\lambda^{k-n-1}}\, y_{n+2} + \frac{k-n}{3}\, \frac{1}{\lambda^{k-n-2}}\, y_{n+3} + \cdots + \\
& + \frac{k-n}{k-n-1}\, \frac{1}{\lambda^2}\, y_{k-1} + \frac{1}{\lambda}\, y_k, \qquad k = n+1, n+2, \cdots.
\end{aligned} \tag{45}$$

For $|\lambda| > 1$, by inserting (45) into (44), rearranging, and using differentiation of the geometric series, we have

$$\begin{aligned}
y_1 = {} & \lambda\, x_1 - \frac{\lambda^2}{(\lambda-1)^2}\left(\frac{\rho}{\lambda^n}\, x_1 + \frac{\rho}{\lambda^n}\, y_2 + \frac{\rho}{\lambda^{n-1}}\, y_3 + \cdots + \frac{\rho}{\lambda^2}\, y_n + \frac{1}{\lambda}\, y_{n+1}\right) - \\
& - \alpha_{n+2}\, y_{n+2} - \alpha_{n+3}\, y_{n+3} - \alpha_{n+4}\, y_{n+4} - \cdots,
\end{aligned} \tag{46}$$

where

$$\alpha_{n+k} = \frac{1}{\lambda} + \frac{k+1}{k}\,\frac{1}{\lambda^2} + \frac{k+2}{k}\,\frac{1}{\lambda^3} + \frac{k+3}{k}\,\frac{1}{\lambda^4} + \cdots\,, \qquad k = 2, 3, 4, \cdots\,,$$

and

$$|\alpha_{n+k}| \le \frac{1}{|\lambda|}\left(1 + \sum_{i=1}^{\infty} \frac{k+i}{k}\,\frac{1}{|\lambda|^i}\right) \le \frac{1}{|\lambda|}\left(1 + \sum_{i=1}^{\infty}(i+1)\,\frac{1}{|\lambda|^i}\right) = \frac{|\lambda|}{(|\lambda|-1)^2}. \tag{47}$$

Solving for (46) for x_1 gives

$$\begin{aligned} x_1 = \frac{1}{q_{n+1}(\lambda)}\Big[&\lambda^{n-2}(\lambda-1)^2\,y_1 + \rho\,y_2 + \rho\,\lambda\,y_3 + \rho\,\lambda^2\,y_4 + \cdots + \rho\,\lambda^{n-2}\,y_n + \lambda^{n-1}\,y_{n+1} + \\ &+ \bar{\alpha}_{n+2}\,y_{n+2} + \bar{\alpha}_{n+3}\,y_{n+3} + \bar{\alpha}_{n+4}\,y_{n+4} + \cdots\Big], \end{aligned} \tag{48}$$

where

$$\bar{\alpha}_{n+k} = \lambda^{n-2}(\lambda-1)^2\,\alpha_{n+k},$$

and

$$|\bar{\alpha}_{n+k}| \le |\lambda|^{n-1}\,\frac{(|\lambda|+1)^2}{(|\lambda|-1)^2}. \tag{49}$$

By inserting (48) into (44) and (45), we have

$$x = (\lambda I - G_n)^{-1} y = \frac{1}{q_{n+1}(\lambda)}\,(A+B)\,y$$

where the matrix representations of A and B are given by

$$\mathbf{A} = \begin{pmatrix}
(\lambda-1)^2\lambda^{n-2} & \rho & \rho\,\lambda & \rho\,\lambda^2 & \cdots & \rho\,\lambda^{n-2} & \lambda^{n-1} & \bar{\alpha}_{n+2} & \bar{\alpha}_{n+3} & \cdots \\
(\lambda-1)^2\lambda^{n-3} & \frac{\rho}{\lambda} & \rho & \rho\,\lambda & \cdots & \rho\,\lambda^{n-3} & \lambda^{n-2} & \frac{\bar{\alpha}_{n+2}}{\lambda} & \frac{\bar{\alpha}_{n+3}}{\lambda} & \cdots \\
(\lambda-1)^2\lambda^{n-4} & \frac{\rho}{\lambda^2} & \frac{\rho}{\lambda} & \rho & \cdots & \rho\,\lambda^{n-4} & \lambda^{n-3} & \frac{\bar{\alpha}_{n+2}}{\lambda^2} & \frac{\bar{\alpha}_{n+3}}{\lambda^2} & \cdots \\
(\lambda-1)^2\lambda^{n-5} & \frac{\rho}{\lambda^3} & \frac{\rho}{\lambda^2} & \frac{\rho}{\lambda} & \cdots & \rho\,\lambda^{n-5} & \lambda^{n-4} & \frac{\bar{\alpha}_{n+2}}{\lambda^3} & \frac{\bar{\alpha}_{n+3}}{\lambda^3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
(\lambda-1)^2\lambda & \frac{\rho}{\lambda^{n-3}} & \frac{\rho}{\lambda^{n-4}} & \frac{\rho}{\lambda^{n-5}} & \cdots & \rho\,\lambda & \lambda^2 & \frac{\bar{\alpha}_{n+2}}{\lambda^{n-3}} & \frac{\bar{\alpha}_{n+3}}{\lambda^{n-3}} & \cdots \\
(\lambda-1)^2 & \frac{\rho}{\lambda^{n-2}} & \frac{\rho}{\lambda^{n-3}} & \frac{\rho}{\lambda^{n-4}} & \cdots & \rho & \lambda & \frac{\bar{\alpha}_{n+2}}{\lambda^{n-2}} & \frac{\bar{\alpha}_{n+3}}{\lambda^{n-2}} & \cdots \\
(\lambda-1)^2\,\frac{1}{\lambda} & \frac{\rho}{\lambda^{n-1}} & \frac{\rho}{\lambda^{n-2}} & \frac{\rho}{\lambda^{n-3}} & \cdots & \frac{\rho}{\lambda} & 1 & \frac{\bar{\alpha}_{n+2}}{\lambda^{n-1}} & \frac{\bar{\alpha}_{n+3}}{\lambda^{n-1}} & \cdots \\
(\lambda-1)^2\,\frac{\rho}{\lambda^2} & \frac{\rho^2}{\lambda^n} & \frac{\rho^2}{\lambda^{n-1}} & \frac{\rho^2}{\lambda^{n-2}} & \cdots & \frac{\rho^2}{\lambda^2} & \frac{\rho}{\lambda} & \frac{\rho\,\bar{\alpha}_{n+2}}{\lambda^n} & \frac{\rho\,\bar{\alpha}_{n+3}}{\lambda^n} & \cdots \\
(\lambda-1)^2\,\frac{2\rho}{\lambda^3} & \frac{2\rho^2}{\lambda^{n+1}} & \frac{2\rho^2}{\lambda^n} & \frac{2\rho^2}{\lambda^{n-1}} & \cdots & \frac{2\rho^2}{\lambda^3} & \frac{2\rho}{\lambda^2} & \frac{2\rho\,\bar{\alpha}_{n+2}}{\lambda^{n+1}} & \frac{2\rho\,\bar{\alpha}_{n+3}}{\lambda^{n+1}} & \cdots \\
(\lambda-1)^2\,\frac{3\rho}{\lambda^4} & \frac{3\rho^2}{\lambda^{n+2}} & \frac{3\rho^2}{\lambda^{n+1}} & \frac{3\rho^2}{\lambda^n} & \cdots & \frac{3\rho^2}{\lambda^4} & \frac{3\rho}{\lambda^3} & \frac{3\rho\,\bar{\alpha}_{n+2}}{\lambda^{n+2}} & \frac{3\rho\,\bar{\alpha}_{n+3}}{\lambda^{n+2}} & \cdots \\
(\lambda-1)^2\,\frac{4\rho}{\lambda^5} & \frac{4\rho^2}{\lambda^{n+3}} & \frac{4\rho^2}{\lambda^{n+2}} & \frac{4\rho^2}{\lambda^{n+1}} & \cdots & \frac{4\rho^2}{\lambda^5} & \frac{4\rho}{\lambda^4} & \frac{4\rho\,\bar{\alpha}_{n+2}}{\lambda^{n+3}} & \frac{4\rho\,\bar{\alpha}_{n+3}}{\lambda^{n+3}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{pmatrix},$$

and

$$
\mathbf{B} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{\lambda} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{\lambda^2} & \frac{1}{\lambda} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{\lambda^3} & \frac{1}{\lambda^2} & \frac{1}{\lambda} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
0 & \frac{1}{\lambda^{n-1}} & \frac{1}{\lambda^{n-2}} & \frac{1}{\lambda^{n-3}} & \frac{1}{\lambda^{n-4}} & \cdots & \frac{1}{\lambda} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{\rho}{\lambda^{n}} & \frac{\rho}{\lambda^{n-1}} & \frac{\rho}{\lambda^{n-2}} & \frac{\rho}{\lambda^{n-3}} & \cdots & \frac{\rho}{\lambda^2} & \frac{1}{\lambda} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{2\,\rho}{\lambda^{n+1}} & \frac{2\,\rho}{\lambda^{n}} & \frac{2\,\rho}{\lambda^{n-1}} & \frac{2\,\rho}{\lambda^{n-2}} & \cdots & \frac{2\,\rho}{\lambda^3} & \frac{2}{\lambda^2} & \frac{1}{\lambda} & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{3\,\rho}{\lambda^{n+2}} & \frac{3\,\rho}{\lambda^{n+1}} & \frac{3\,\rho}{\lambda^{n}} & \frac{3\,\rho}{\lambda^{n-1}} & \cdots & \frac{3\,\rho}{\lambda^4} & \frac{3}{\lambda^3} & \frac{3}{2\,\lambda^2} & \frac{1}{\lambda} & 0 & 0 & 0 & \cdots \\
0 & \frac{4\,\rho}{\lambda^{n+3}} & \frac{4\,\rho}{\lambda^{n+2}} & \frac{4\,\rho}{\lambda^{n+1}} & \frac{4\,\rho}{\lambda^{n}} & \cdots & \frac{4\,\rho}{\lambda^5} & \frac{4}{\lambda^4} & \frac{4}{2\,\lambda^3} & \frac{4}{3\,\lambda^2} & \frac{1}{\lambda} & 0 & 0 & \cdots \\
0 & \frac{5\,\rho}{\lambda^{n+4}} & \frac{5\,\rho}{\lambda^{n+3}} & \frac{5\,\rho}{\lambda^{n+2}} & \frac{5\,\rho}{\lambda^{n+1}} & \cdots & \frac{5\,\rho}{\lambda^6} & \frac{5}{\lambda^5} & \frac{5}{2\,\lambda^4} & \frac{5}{3\,\lambda^3} & \frac{5}{4\,\lambda^2} & \frac{1}{\lambda} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{pmatrix},
$$

respectively. For $|\lambda| > 1$, by using differentiation of the geometric series, (49), and the argument used in (47), we have $\|A\|_1 \leq \infty$ and $\|B\|_1 \leq \infty$. Thus, for $|\lambda| > 1$ and $q_{n+1}(\lambda) \neq 0$, the operator $\lambda I - G_n$ has a bounded inverse and its resolvent set is given by (33).

Step 3. The point spectrum of the transposed operator G_n^t consists of all $\lambda \in \mathbb{R}$, $|\lambda| \leq 1$, such that

$$
(\lambda I - G_n^t)\, x = 0, \qquad x \neq 0, \quad \|x\|_\infty < \infty.
$$

This is equivalent to

$$
\begin{aligned}
x_k &= \lambda^{k-1}\, x_1, \qquad k = 2, \cdots, n, \\
x_{n+1} &= \frac{\lambda^n}{\rho}\, x_1, \\
x_{n+k} &= \frac{1}{k}\left(\frac{\lambda^{n+k-1}}{\rho} - \lambda^{k-2} - 2\,\lambda^{k-3} - 3\,\lambda^{k-4} - \cdots - (k-2)\,\lambda - (k-1)\right) x_1, \quad k \geq 2.
\end{aligned}
$$

If $\lambda \neq 1$, we have

$$
\begin{aligned}
x_{n+k} &= \frac{1}{k}\left[\frac{\lambda^{n+k-1}}{\rho} - \left(\lambda^{k-2} + 2\,\lambda^{k-3} + \cdots + (k-2)\,\lambda + (k-1)\right)\frac{(\lambda-1)^2}{(\lambda-1)^2}\right] x_1 \\
&= \frac{1}{k}\left(\frac{\lambda^{n+k-1}}{\rho} - \frac{\lambda^k - k\,\lambda + (k-1)}{(\lambda-1)^2}\right) x_1, \quad k \geq 2.
\end{aligned}
$$

Therefore

$$
|x_{n+k}| \leq \frac{1}{k}\left(\frac{1}{\rho} + \frac{1+k+k-1}{|\lambda-1|^2}\right) x_1 \leq \left(\frac{1}{2\,\rho} + \frac{2}{|\lambda-1|^2}\right) x_1, \qquad k \geq 2,
$$

which implies $\|x\|_\infty < \infty$. For $\lambda = 1$ we have

$$
\begin{aligned}
x_{n+k} &= \frac{1}{k}\left(\frac{1}{\rho} - 1 - 2 - 3 - \cdots - (k-2) - (k-1)\right) x_1 \\
&= \frac{1}{k}\left(\frac{1}{\rho} - \frac{(k-1)\,k}{2}\right) x_1, \qquad k \geq 2,
\end{aligned}
$$

so $\|x\|_\infty = \infty$. We conclude that the point spectrum of G_n^t is given by (34). This, in turn, implies (35) and (36) as described before.

Example 2 The lesion forming plant pathogen potato late blight (phytophthora infestans) grows radially on a leaf with a constant daily rate. The latency period for a lesion to become infectious is five days, and the sporulating area is infectious for one day. In [PSV05] the epidemic spread of such pathogen is modeled with the infinite dimensional Leslie matrix of the form of Γ_5 as defined in (27). Further, the upper bound for the speed of invasion in computed via minimization of the largest eigenvalue $\lambda_{\max}(\Gamma_5(s))$. From Remark 2 it follows that this eigenvalue is the largest unique positive root of $q_6(\lambda)$ from (29). Here the parameter ρ has the form $\rho(s) = \text{const} \times M(s)$ where $M(s)$ is some moment-generating function (for example, $M(s) = \exp(\sigma^2 s^2/2)$ for the Gaussian kernel or $M(s) = 1/(1-\sigma^2 s^2)$ for the Laplace kernel). Here $\Gamma_5(s)$ appears naturally due to the fact that the considered pathogen has a latency period of five days. It is interesting that $\lambda_{\max}(\Gamma_5(s))$ can be computed analytically:

$$\lambda_{\max}(\Gamma_5) = \frac{1}{3} + \frac{2^{1/3}}{3\left(2+27\sqrt{\rho}+\sqrt{108\sqrt{\rho}+729\rho}\right)^{1/3}} + \frac{\left(2+27\sqrt{\rho}+\sqrt{108\sqrt{\rho}+729\rho}\right)^{1/3}}{3\cdot 2^{1/3}}.$$

The speed of invasion is bounded by

$$v^* = \min_{0<s<\hat{s}} \frac{1}{s} \ln\left[\lambda_{\max}(\Gamma_5(s))\right],$$

where $\hat{s}$ is the maximum s for which $M(s)$ is defined. For details about a rather complex derivation of this model we refer the reader to [PSV05].

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